Chapter 6 Derivatives

How can we find how steep a graph is?

Of course we know that if the slope of a linear equation is, for instance, positive and large, the line will rise steeply from left to right. But for any function, how can we find the steepness of its graph?
The steepness of a curve at a particular point can be defined as the slope of the tangent (a straight line that touches the curve at a certain point) to the curve at that specific point.
The slope of the tangent to a curve at a particular point is the called the **derivative.** The notation of the derivative is: $f'(x)$.

Formulas for finding the derivative and tangent of a function:

Newton Quotient (at point a):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Ex: Find $f'(x)$ when $f(x) = 4x^2$, using the definition (Newton quotient) of derivatives.

$$= \frac{4(x+h)^2 - 4(x)^2}{h}$$

$$= \frac{8hx + 4h^2}{h}$$

$= 8x + 4h$ Since h approaches 0 (thus $4h$ will approach 0), the derivative will simply be:

$f'(x) = 8x$

Definition of a tangent (for the graph of $y = f(x)$ at the point $(a, f(a))$

$y - f(a) = f'(a)(x - a)$

To find the tangent, you do the following:

1. Find the derivative.
2. Then substitute the point (that's given) in $y - f(a) = f'(a)(x - a)$

Finally, there are of course different ways of writing the derivative:

$$\frac{dy}{dx} \text{ or } \frac{df(x)}{dx} \text{ or } \frac{d}{dx} f(x)$$

Increasing or decreasing:

$f'(x) > 0, f \text{ will increase}$
$f'(x) < 0, f \text{ will decrease}$
$f'(x) = 0, f \text{ is constant}$

This makes sense since the derivative represent the slope of a function, and thus if a slope is for instance 2, the line will rise from left to right- the function is increasing.
Ex. Examine where $f(x) = 2x^2 - 2$ is increasing/decreasing.
Step 1: Find the derivative: $f'(x) = 4x$.
Step 2: Then you set the derivative equal to zero, and solve for $x \longrightarrow$ $0 = 4x$, thus $x = 0$.

Step 3: Then you substitute a value that is lower than 0 (for instance -2) into the first derivate, and a value that is higher than 0 (for instance 2), this will allow you to find the intervals for which the function will be increasing or decreasing.

$f'(x) = 4(-2)$	$f'(x) = 4(2)$
-8 = decreasing	8= increasing

Answer: $f(x)$ is increasing when: $f(x) > 0$, and decreasing when $f(x) < 0$.

The derivative is a rate as well:

A derivative is actually just a rate- it measure how much something changes compared to another.
One way of remembering the connection between slope and rate is the following:

$$slope = \frac{rise}{run} = \frac{dy}{dx} = rate$$

The slope of the tangent line at a particular point is the **instantaneous rate**, and therefore the instantaneous rate of change of f at a is $f'(a)$.

Secondly, the **relative rate of change** of f at a is $\frac{f'(a)}{f(a)}$, this can be used to describe, for instance, how much a variable changed this year (written as a percentage).

Small introduction to limits:

Rules for limits:
If $\lim_{x\to c} f(x) = C$ and $\lim_{x\to d} g(x) = D$, then:

1) $\lim_{x\to c}\big(f(x) \pm g(x)\big) = C \pm D$
2) $\lim_{x\to c}\big(f(x) \cdot g(x)\big) = C \cdot D$
3) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{C}{D}$, if $D \neq 0$
4) $\lim_{x\to c}(f(x))^r = A^r$ (if A^r is defined and r is any real number)

Differentiation
Differentiation is the process of finding a derivative.

Differentiation methods:

Power rule:
$f(x) = x^a \to f'(x) = ax^{a-1}$
Sum and difference:
$F(x) = f(x) \pm g(x) \to F'(x) = f'(x) \pm g'(x)$
Product rule:
$F(x) = f(x) \cdot g(x) \to F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Ex. Find $F'(x)$, when $F(x) = (x^2 - 1) \cdot (x + 4)$.
$f(x) = (x^2 - 1)$ $\qquad f'(x) = 2x$ (using the power rule)

$g(x) = (x + 4)$ $g'(x) = x$ (using the power rule)

$F'(x) = (2x)(x + 4) + (x^2 - 1)(x)$

Answer: $F'(x) = x^3 + 2x^2 + 7x$

Quotient:

$$F'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2}$$

Ex. Find $F'(x)$ when $F(x) = \frac{2x^2+1}{4x-2}$.

$f(x) = 2x^2 + 1$ $f'(x) = 4x$

$g(x) = 4x - 2$ $g'(x) = 4$

$$F'(x) = \frac{4x \cdot (4x - 2) - (2x^2 + 1) \cdot 4}{(4x - 2)^2}$$

Chain Rule:

Officially the chain rule is written as: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ or $F'(x_0) = f'(u_0)g'(x_0)$

However, the chain rule can also be described the following way:

$y = u^a$, $y' = au^{a-1} \cdot u'$, (where u is usually the expression in brackets).

Ex. Find $F'(x)$ when $F(x) = 3(x^2 - 1)^2$.

In this case $u = (x^2 - 1)^2$ *and thus you find:* $F'(x) = 6(x^2 - 1) \cdot 2x$

Other order derivatives:

$f'(x) = \textit{first derivative}$

$f''(x) = \textit{second derivative}$

Ex. Find $f'(x)$ and $f''(x)$ if $f(x) = 4x^3 - 2x + 1$

$f'(x) = 12x^2 - 2$

$f''(x) = 24x$ (simply find the derivative of the first derivative)

Convex or concave:

1) f *is convex on the interval if* $f''(x) \geq 0$ *for all* x'*s on the interval*

2) f *is concave on the interval if* $f''(x) \leq 0$ *for all* x'*s on the interval*

Note on concavity:

If a function concaves down its derivative is decreasing, and when a function concaves up its derivative is increasing. If the concavity switches at a point it is point of inflection. (however this will be discussed later)

The difference between convex and concave will be important when using economic models!

N^{th} *order:*

As mentioned earlier, we can find the second derivative, but we can also find for instance the tenth derivative; the n^{th} derivative is written the following way: $y^n = f^n(x)$.

Exponential Functions and Differentiation

The derivative of e^x*:*

$f(x) = e^x \rightarrow f'(x) = e^x$

However, if you have to find for instance the derivative of $f(x) = e^{x^2}$, the following rule applies:

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$f(x) = e^u \rightarrow f'(x) = u' \cdot e^u$

Logarithmic functions and differentiation:
The derivative of lnx:

$f(x) = lnx \rightarrow f'(x) = \frac{1}{x}$

Ex. Find $f'(x)$ when $f(x) = lnx + x^2$

$f'(x) = \frac{1}{x} + 2x$

Chapter 7 Application of Derivatives:

Application of Derivatives:

Implicit Differentiation:

1. Implicit Differentiation is used when the function is not in the form a= f(b), but the variables x and y could be on either side of the equation,
2. Instead of getting the equations to the form a=f(b), just take derivative on both sides (REMEMBER: re-write 'a' as f(b) considering 'a' to be a function of 'b')
3. Solve the equation for y'

Example:

$2a^2 + 4ab = 2b^2$

$2a^2 + 4af(b) = 2[f(a)]^2$

(Remember: use chain and product rule where applicable)

$2 \times 2 \times a + 4a^0 f(a) + 4af'(a) = 2 \times 2 \times f(a) \times f'(a)$

$4a + 4f(a) + 4af'(a) = 4f(a)f'(a)$

$4a + 4f(a) = f'(a)[4f(a) - 4a]$

$f'(a) = \frac{4a + 4f(a)}{4f(a) - 4a} = \frac{4a + 4b}{4b - 4a} = b'$

Inverse and Differentiation

If a function is differentiable, and it is strictly increasing or decreasing then for that interval function will have an inverse.

$for\ a\ function\ b = f(a), \qquad h'(b_i) = \frac{1}{f'(a_i)}$

For Example: $Find\ the\ inverse\ of\ the\ equation\ f(x) = e^x$

$y = e^x = f(x)$

$x = g(y) = lny$

$g'(y) = \frac{1}{f'(x)}\ from\ the\ equation\ given\ above$